海洋科学技術センター試験研究報告　第36号　JAMSTECR, 36 (September 1997)

拡散係数が時間及び空間の関数である場合の１解析解

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拡散係数が時間及び空間の関数である場合の減衰項を含む拡散方程式の解析解を求めた。ここで拡散係数は空間的には片側の境界からある距離まで距離に比例して増加し，また時間的にはt＝０からある時間まで時間に比例して増加する。解は固有関数展開により求めたが，その結果，もとの偏微分方程式を解く問題は１変数の積分方程式を解く問題に変換され，この積分方程式を数値的に解いた。

キーワード：拡散方程式

One analytical solution of a diffusion equation when diffusivity is a function of time and space

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An analytical solution for a diffusion equation with a decay term when the diffusion coefficient is a function of time and space is obtained.
The diffusion coefficient increases linearly from one boundary to a certain distance from that boundary,and it becomes independent of space beyond that point.The diffusion coefficient also linearly increases with time from t=0, but it becomes independent of time after a certain period has passed. The solution is obtained by the method of eigenfunction expansion and the original problem of solving a partial differential equation is transformed into a problem of solving an integral equation with a single variable. This integral equation is solved numerically.

Key Words : Diffusion

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1 Introduction

Due to the development of computers, it becomes fairly common for scientists to solve differential equations numerically in recent years. Analytical solutions are not necessarily always available and even if they are obtained, forms of solutions may be too complicated to be useful. Nevertheless, it is generally true that analytical solutions give us some useful information regarding the natures of those solutions such as their dependencies on parameters without performing extensive computations. It also is true that, in many cases,numerical evaluations of analytical solutions require much less computational resources and programming efforts. The solution described here was originally obtained for the author's research related to a dispersion problem of the larvae of southern bluefin tuna (Matsuura *et al.*, 1997)[1]. In Matsuura et al. (1997)[1], only the basic equations and solutions are shown ; however,the solution may have some other applications as well and some details are described here.

2 Equations

The equation of one dimensional diffusion treated here is

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=\frac{\partial}{\partial y}(K\frac{\partial C}{\partial y})-\alpha C \qquad (1)$$

where, C is the concentration, u the advection speed which is a constant, t the time, K the horizontal eddy diffusion coefficient normal to the boundary and α is the decay factor. Here, only the diffusion in y direction is considered. The advection in x direction (second term of left hand side) is not essential if it is a constant for the case such as above in which diffusion in x direction is not considered. Nevertheless, for the sake of consistency with the original paper, it is included here. In the following text, α is taken as a constant or the function of time alone. The diffusion coefficient, K, is a function of time and space and

$K=\kappa$ ty for $0 \le y < L$ and $0 \le t < to$

$K=\kappa$ tL for $L < y \le M$ and $0 \le t < to$

$K=\kappa$ t_0y for $0 \le y < L$ and $to < t$

$K=\kappa$ toL=Constant for $L < y \le M$ and $to < t$ (2)

where κ is a constant, L the distance from one side of a boundary where the effect of that boundary vanishes and M is the width of the domain. Note that K is continuous at $t=to$ and $y=L$. This form of K reproduces linear increase of the diffusion coefficient in time during initial period as observed elsewhere (Poulain and Niller[2], 1989 ; Matsuura *et al.*, 1997[1]) and linear increase of the component of diffusion coefficient normal to the coast (boundary) near the coast (Davis, 1985)[3]. No flux condition ($K\ \partial C/\partial y=0$) is applied at both boundaries ; i. e. at $y=0$ and at $y=M$. As a matching condition,both C and $K\ \partial C/\partial y$ are continuous at $y=L$.

The solution of (1) for the observer moving with an advection speed u, whose position is x=ut, is

$$C=e^{(-\int_0^t \alpha d\tau)}[\kappa\int_0^t \tau p\ d\tau+\sum_{n=1}^{\infty}\{\frac{\kappa}{Jo(2\sqrt{\lambda_n L})}\int_0^t \tau p e^{\kappa\lambda_n(\tau^2-t^2)/2}d\tau\ \phi_n\}]$$

for $0 \le y < L$ *and* $0 \le t < to$

$$C=e^{(-\int_0^t \alpha d\tau)}[-\frac{\kappa L}{M-L}\int_0^t \tau p\ d\tau+\frac{1}{M-L}\int_L^M Co\ dy+\sum_{n=1}^{\infty}\{[-\frac{2\kappa L}{M-L}\int_0^t \tau p e^{\kappa L\gamma_n(\tau^2-t^2)/2}d\tau+\frac{2}{M-L}e^{-\kappa L\gamma_n t^2/2}\int_L^M Co\ \Phi n\ dy]\Phi n\}]$$

for $L < y \le M$ *and* $0 \le t < to$

$$C=e^{(-\int_0^t \alpha\ d\tau)}[\kappa\int_{t_0}^t p\ d\tau+a_0+\sum_{n=1}^{\infty}\{\frac{\kappa}{Jo(2\sqrt{\lambda nL})}\int_{t_0}^t p e^{\kappa\lambda_n(\tau-t)}d\tau+a_n e^{-\kappa\lambda_n(t-to)}]\phi_n\}]$$

for $0 \le y < L$ *and* $to < t$

$$C=e^{(-\int_0^t \alpha\ d\tau)}[-\frac{\kappa L}{M-L}\int_{t_0}^t p\ d\tau+b_0+\sum_{n=1}^{\infty}\{[-\frac{2\kappa L}{M-L}\int_{t_0}^t p e^{\kappa L\gamma_n(\tau-t)}d\tau+b_n e^{-\kappa L\gamma_n(t-to)}]\Phi n\}]$$

$$for\ L < y \leq M\ and\ to < t \quad (3)$$

where

$$\phi_n = Jo(2\sqrt{\lambda_n y})$$

$$\Phi n = \cos\{\sqrt{\gamma_n}(y-L)\} \qquad \gamma_n = (\frac{n\pi}{M-L})^2$$

$$a_0 = \kappa \int_0^{t0} \tau p\ d\tau$$

$$a_n = \frac{\kappa}{Jo(2\sqrt{\lambda nL}} \int_0^{t_0} \tau p e^{\kappa\lambda n(\tau^2 - t_0^2)/2} d\tau$$

$$b_0 = -\frac{\kappa L}{M-L} \int_0^{t_0} \tau p\ d\tau + \frac{1}{M-L} \int_L^M Co\ dy$$

$$b_n = -\frac{2\kappa L}{M-L} \int_0^{t_0} \tau p e^{\kappa L\gamma_n(\tau^2 - t_0^2)/2} d\tau$$

$$+\frac{2}{M-L} e^{-\kappa L\gamma_n t_0^2/2} \int_L^M Co\ \Phi n\ dy,$$

λ_n, an n-th eigenvalue which satisfies $J_1(2\sqrt{\lambda nL}) = 0$, Jo the Bessel function of order of 0 and $J1$ is the Bessel function of order of 1. Co is the initial distribution of C and is assumed to be zero for $0 \leq y < L$. This assumption was necessary in the original work (Matsuura *et al*, 1997)[1]. The effect of α, if it is a positive constant value such as in the case of radio active decay, is that the concentration would decrease exponentially with the elapsed time and the value of α^{-1} itself represents the time scale of the reduction of C. This solution is obtained by the method of eigenfuction (ϕn and Φn) expansion. Since diffusion coefficient, K, becomes 0 at $y=0$, no flux condition at that boundary does not guarantee $\partial C/\partial y=0$ at $y=0$ while $\partial C/\partial y$ is 0 at $y=M$. In the process of obtaining the eigenfunction for $0 \leq y < L$, change of variable, $2\sqrt{\lambda_n y} -> \xi$, was used. After this transformation, original Sturm-Liouville problem,

$$\frac{d}{dy}(y\frac{d}{dy}\phi_n) = -\lambda n\phi_n,\ \frac{d}{dy}\phi_n = 0$$

$$at\ y = 0\ and\ y = L$$

becomes Bessell's differential equation with homogeneous boundary conditions.Following relations are also used to obtain (3)

$$\int_0^L \Phi n^2 dy = L\ J_0^2(2\sqrt{\lambda nL}),$$

$$\int_0^L \Phi n\ dy = 0$$

These relations may be obtained directly from original Sturm-Liouville equations without performing change of variable. The terms with C_0 can be simplified if gaussian distribution is selected for C_0. In that case, with the conditions that $\int_L^M C\ dy=1$ (unit volume) and half amplitude width is $2W$, C_0 is,using the contour integration,

$$C_0 = e^{-r(\mathcal{Y}-\mathcal{Y}_0)^2} / [\sqrt{\frac{\pi}{r}} \{1 - \frac{1}{2}[erfc(\sqrt{r}$$

$$(\mathcal{L}-\mathcal{Y}_0)) + erfc(\sqrt{r}\mathcal{Y}_0)]\}]$$

where $\mathcal{Y} = y-L$, $\mathcal{L} = M-L$, $r = -\frac{1}{W^2} ln(\frac{1}{2})$, $\mathcal{Y}_0$ the position of the center of the distribution measured from L, and erfc is the complementary error function. Note that the assumtion is $Co=0$ for $0 \leq y < L$. If W is small enough relative to $\mathcal{L}$, the denominator may be approximated by $\sqrt{\frac{\pi}{r}}$. From this C_0,

$$\int_L^M Co\ \Phi_k\ dy = e^{-\frac{1}{4r}\gamma_k} \cos(\sqrt{\gamma_k}\mathcal{Y}_0)$$

Using aforementioned eigenfunctions, solution shown above is expressed as a function of P, which is proportional to the flux at $y=L$ and defined as

$$p \equiv e^{(\int_0^t \alpha\ d\tau)} \frac{\partial C}{\partial y}|_{y=L}$$

It is noted here that all the modes are connected through this function. This is because eigenfunctions for $0 \leq y < L$ and for $L < y \leq M$ are different and thus flux at the boundary must be decomposed into each mode in different ways in each domain at the boundary.

From the matching condition, a function P satisfies

$$\kappa \sum_{n=1}^{\infty} \{\int_0^t \tau p e^{\kappa\lambda n(\tau^2 - t^2)/2} d\tau\} + \frac{2\kappa L}{M-L} \sum_{n=1}^{\infty}$$

$$\{\int_0^t [\tau p e^{\kappa L\gamma_n(\tau^2 - t^2)/2}] d\tau\} + \frac{\kappa M}{M-L} \int_0^t \tau p\ d\tau$$

$$= \frac{1}{M-L} \int_L^M Co\ dy + \frac{2}{M-L} \sum_{n=1}^{\infty}$$

$$\{e^{-\kappa L\gamma_n t^2/2} \int_L^M Co\ \Phi n\ dy\}\ for\ 0 \leq t < t_0$$

$$\kappa \sum_{n=1}^{\infty} \{[\int_{t_0}^{t} pe^{\kappa\lambda n(\tau-t)}d\tau+\frac{2\kappa L}{M-L}\sum_{n=1}^{\infty}$$

$$\{\int_{t_0}^{t}[pe^{\kappa L\gamma_n(\tau-t)}]d\tau\}+\frac{\kappa M}{M-L}\int_{t_0}^{t} p\ d\tau$$

$$= \{b_0+\sum_{n=1}^{\infty}(b_n e^{-\kappa L\gamma_n(t-t_0)})\}-\{a_0+$$

$$\sum_{n=1}^{\infty}(\frac{a_n}{Jo(2\sqrt{\lambda_n L})}e^{-\kappa\lambda_n(t-t_0)})\}$$

$$for\ t_0 < t \qquad (4)$$

Inspection of (4) reveals that the unknown function P is included in the integration of the form,

$$\int_0^t p(\tau)Q(\tau)d\tau \qquad (5)$$

where Q is a known function. Thus, original problem of solving a partial differential equation becomes a problem solving an integral equation of P. We approximate above integration as

$$\int_0^t p(\tau)Q(\tau)d\tau \approx \Delta t\{p\,|_{\tau=t-\Delta t}^{\tau=t} Q\,|_{\tau=t-\Delta t}^{\tau=t} + p\,|_{\tau=t-2\Delta t}^{\tau=t-\Delta t} Q\,|_{\tau=t-2\Delta t}^{\tau=t-\Delta t} + \ldots + p\,|_{\tau=0}^{\tau=\Delta t} Q\,|_{\tau=0}^{\tau=\Delta t}\} \qquad (6)$$

where $Q\,|_{\tau=t-\Delta t}^{\tau=t}$ indicates average of Q between $\tau = t$ and t-Δt

Appling (6) to (4) yields,

$$p(n\Delta t)[\sum_{k=1}^{\infty}\{\frac{1}{\lambda_k}[1-e^{-\kappa\lambda_k\Delta t^2(2n-1)/2}]\}+$$

$$\frac{2}{M-L}\sum_{k=1}^{\infty}\{\frac{1}{\gamma_k}[1-e^{-\kappa L\gamma_k\Delta t^2(2n-1)/2}]\}+$$

$$\frac{\kappa M}{M-L}\frac{\Delta t^2}{2}(2n-1)] = \frac{1}{M-L}\int_L^M Co\ dy+$$

$$\frac{2}{M-L}\sum_{k=1}^{\infty}\{e^{-\kappa L\gamma_k(n\Delta t)^2/2}\int_L^M Co\ \Phi n\ dy\}-$$

$$\sum_{k=1}^{\infty}[\frac{1}{\lambda_k}\sum_{j=1}^{n-1}\{p((n-j)\Delta t)e^{-\kappa\lambda_k\Delta t^2 j(2n-j)/2}$$

$$[1-e^{-\kappa\lambda_k\Delta t^2(2(n-j)-1)/2}]\}]-$$

$$\frac{2}{M-L}\sum_{k=1}^{\infty}[\frac{1}{\gamma_k}\sum_{j=1}^{n-1}\{p((n-j)\Delta t)e^{-\kappa L\gamma_k\Delta t^2 j(2n-j)/2}$$

$$[1-e^{-\kappa L\gamma_k\Delta t^2(2(n-j)-1)/2}]\}]-$$

$$\frac{\kappa M}{M-L}\frac{\Delta t^2}{2}\sum_{j=1}^{n-1}[p((n-j)\Delta t)\{2(n-j)-1\}] \qquad (7)$$

where $p(n\Delta t) \equiv p\,|_{\tau=(n-1)\Delta t}^{\tau=n\Delta t}$

Using (7), we can calculate P progressively with the condition that $P(0)=0$ (No flux at $t=0$). This initial condition was valid for the case used in original work (Matsuura *et al*, 1997)[1] within computational accuracy. Note that Δt must be small enough for the approximation (6) be valid. Appropriate Δt may be chosen by considering the time scale of Q which depends on the mode. The formula for $t_0 < t$ is similar to (7) and I omit it here.

Equations (3) and (4) indicate effect of the decay factor can be included posteori as long as it is a constant or a function of time alone. For example, instead of choosing a constant, decaying rate may be modeled as a function of time such as $\alpha = \beta + \varepsilon\ exp\ (-\Theta t)$ for an application to a biological problem. By this formulation, decaying rate is $\beta + \varepsilon$ at $t=0$ but approaches to β exponentially as time increases. The modification of the solution for this form of decaying rate from the constant decaying rate is rather trivial ; substitute exp $(-\beta t + \varepsilon/\Theta\ \exp(-\Theta t))$ into the first exponential terms of the solutions.

3 Result

Solution was computed for the case when M is *1* 700km, L 100km, to 150 hours, $\kappa\ t_0 L$ $3.6\times10^3 m^2/s$ and W is 50km as a standard case. These values are chosen in Matsuura *et al.* (1997)[1] based on observations of tracks of drifting buoys.

We can approximate (7) by truncating it at certain mode since the contribution from higher mode (larger k) decreases as mode number increases (because both λ_k and γ_k increase).Although I have computed up to 200th mode,test computation with lesser terms indicates 200 modes are more than enough. However, I did not pursue any more test to get minimun terms necessary to compute (7) with reasonable accuracy provided that such a trial would take too much time since the computers convenient to use at that time were personal computers with Pentium (75 Mhz) and Power*Pc* 604 (150Mhz) Cpus. Note that if the initial distribution is highly concentrated, larger number of terms are necessary to reproduce it. In the extreme case such as the case when initial distribution becomes a delta function, infinite number of terms are required.I terminated summation with j as an index (integration in time)

in the third term of right hand side of (7) to reduce computational time, if the condition

$$\left| \frac{\sum_{j=1}^{m-1} \{p((n-j)\Delta t) e^{-\kappa\lambda_k \Delta t^2 j(2n-j)/2}[1-e^{-\kappa\lambda_k \Delta t^2(2(n-j)-1)/2}]\}}{p((m-j)\Delta t) e^{-\kappa\lambda_k \Delta t^2 j(2m-j)/2}[1-e^{-\kappa\lambda_k \Delta t^2(2(m-j)-1)/2}]} \right| > 1 \times 10^{15}$$

continues successively for 20 times. I used double precision for all of the computation and 15 is the approximate significant digit for the double precision (Technically, it is possible to have much larger significant digit by dividing numbers into several segments.). Note that contributions to this summation from terms with larger j are generally smaller due to the term, $e^{-\kappa\lambda_k \Delta t^2 j(2n-j)/2}$. Same measure was taken to compute the fourth term of the right hand side of (7). The time step was 10 minites for $0 \leq t < t_0$ and 20 minites for $t_0 < t$.

The method used here (method of eigenfunction expansion) does not allow term by term differentiation in term of y but it does allow term by term integration in term of y which means it is not necessary to compute C to evaluate $\int_0^t C\,dy$ for a constant l. Therefore, I calculated $\int_0^t \phi_k\,dy$ prior to the computation of (4), and then calculated P, C (to test the results) and $\int_0^t C\,dy$ simultaneously since (7) includes almost all the terms necessary to compute these values.

Figure 1 shows the time series of C integrated from $y=0$ to $y=10$km for the standard case ((i), thick line), for the case dispersion coefficient is 10 time larger ((ii), solid line with solid circles), for the case dispersion is half of the standard case ((iii), dash line), and for the case when the width of the half amplitude point is 200km with the standard value of dispersion coefficient ((iv), fine line), respectively. For the purpose of comparison, the result for the case when dispersion coefficient is constant in space and time is also shown in this figure as a dot line (v). The decay factor, α, is set to 0 for all of the cases. Comparison between case (i) and (v) shows there is a considerable difference of accumulated concentration near the boundary. Case (iv) represents the case when initial

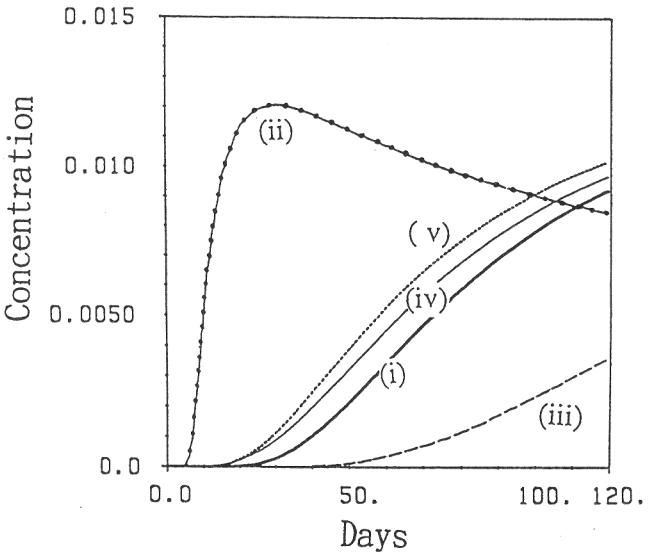

Fig. 1 (a) : Concentration integrated from 10km offshore to the coast as a function of time for an observer moving with an advection speed for the standard case ((i) ; solid curve), for the case dispersion coefficient is 10 time larger ((ii) ;solid line with solid circles),for the case dispersion is half of the standard case ((iii), dash line),and for the case when the width of the half amplitude point is 200 km with the standard value of dispersion coefficient ((iv) ; fine line), respectively. For the case when dispersion coefficient is constant is shown as dot line (v).

distribution is "broader" than the standard case. Case (ii) shows that concentration start decreasing at about 23days. This is caused by the existence of the boundary at $y=0$ and beyond of that y position, particles can not be dispersed. The final value, when the concentration becomes uniform is about 0.0058, and all the cases shown here overshoot this value.

4 Conclusion

An analytical solution of a diffusion equation with a decay term when diffusion coefficient is a function of time and space is obtained.The diffusion coefficient has a form that it increases linearly in time during initial period, and, within a certain distance from one boundary, it also increases linearly as the distance increases from that boundary. The solution is obtained by the method of

eigenfunction expansion and it transformed an original problem of solving a partial differential equation into a problem of solving an integral equation. It shows that flux term at the boundary between two domains connects all the modes. The solution is evaluated by numerically solving this integral equation and compared with the case when diffusion coefficient is a constant. They indicate there is a considerable difference of accumulated concentration near the boundary for the values of parameters chosen here.

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（原稿受理：1997年5月27日）